IN PRAISE OF A LOGIC OF DEFINITIONS THAT TOLERATES Ω-INCONSISTENCY

Anil Gupta
University of Pittsburgh

Abstract

I argue that a general logic of definitions must tolerate ω-inconsistency. I present a semantical scheme, **S**, under which some definitions imply ω-inconsistent sets of sentences. I draw attention to attractive features of this scheme, and I argue that **S** yields the minimal general logic of definitions. I conclude that any acceptable general logic should permit definitions that generate ω-inconsistency. This conclusion gains support from the application of **S** to the theory of truth.

Keywords Circular definitions, revision theory, truth, paradox, McGee's Theorem, omega-inconsistent theories.

The logic of definitions with which I shall be concerned is *general*, in the sense that it countenances systems of interdependent definitions, including systems containing circular definitions.[1] Such a logic *tolerates ω-inconsistency* iff it permits systems of definitions in the language of arithmetic that imply, by the lights of the logic, ω-inconsistent sets of sentences. Recall that a set of sentences of an arithmetical language is *ω-inconsistent* iff it contains a sentence of the form $\exists x \neg F(x)$ as well as the sentences $F(\overline{n})$, for all numerals $\overline{n}$.

Nuel Belnap and I presented in our book, *Revision Theory of Truth* (henceforth: *RTT*), several different semantical schemes for making sense of interdependent definitions, principal among them being $\mathbf{S}^*$ and $\mathbf{S}^\#$. My aim in this paper is to draw attention to a scheme, **S**, that possesses a combination of virtues not found in other schemes and, in particular, not found in $\mathbf{S}^*$ and $\mathbf{S}^\#$.[2] I outline scheme **S** in §1 and apply it to some circular definitions in §2. I go on to draw attention to some of the features of **S** in §3, and I compare it to other schemes in §4. I argue in §5 that **S** yields the minimal general logic of definitions. Since **S** tolerates ω-inconsistency, I conclude that any satisfactory general logic of definitions must do so. I show in §6 that this conclusion gains support from the application of the logic to the theory of truth.

1 Implication in S

I begin with a semantical characterization of the notion of implication in **S**. Let $\mathscr{L}$ be a classical first-order language with identity.[3] Let M $(= \langle D, I\rangle)$ be a model of $\mathscr{L}$, where D is *the domain of* M and I is *the interpretation function of* M. I assigns to a one-place predicate of $\mathscr{L}$, for example, a subset of D. Let $\mathscr{L}$ be extended to a language $\mathscr{L}^+$ through the addition of a system of definitions. For notational simplicity, let us suppose that this system consists of just one definition $\mathscr{D}$:

($\mathscr{D}$) $Gx =_{Df} A(x, G)$.[4]

Here, *the defined term* is a one-place predicate, G, and $A(x, G)$ is *the definiens*. The definiens may contain occurrences of G, rendering the definition circular, but it may not contain any variables free other than x. We shall call $\mathscr{L}$ *the ground language* and M a *ground model of* $\mathscr{L}^+$. Our goal is to define the notion "relative to $\mathscr{D}$, premisses $A_1, \ldots, A_n$ imply in **S** conclusion C" (alternatively, "$A_1, \ldots, A_n$ $\mathscr{D}$-imply C in **S**"; notation: $A_1, \ldots, A_n \models^{\mathscr{D}} C$).[5] We can gain this notion if we can define "sentence B is $\mathscr{D}$-valid in **S** relative to ground model M" (alternatively, "M $\mathscr{D}$-validates B in **S**"; notation: $M \models^{\mathscr{D}} B$). For we can recover the notion of implication in the following way. First we define an absolute notion of $\mathscr{D}$-validity:

(1) B is *$\mathscr{D}$-valid in* **S** (notation: $\models^{\mathscr{D}} B$) iff, for all ground models M of $\mathscr{L}^+$, $M \models^{\mathscr{D}} B$.

Then we define "implication" thus:

(2) $A_1, \ldots, A_n \models^{\mathscr{D}} C$ iff $\models^{\mathscr{D}} [(A_1 \mathbin{\&} \ldots \mathbin{\&} A_n) \supset C]$.

So, the task before us is to define "$M \models^{\mathscr{D}} B$." We shall do this using revision-theoretic ideas, which we now review. (For motivation and fuller explanation, see *RTT*, chapters 4 and 5.)

Given a ground model M, subsets of the domain D are possible *hypotheses* concerning the extension of G. If h is one such hypothesis, then let $M + h$ be the model of $\mathscr{L}^+$ that is just like M except that it assigns to G the interpretation h. Now, definition $\mathscr{D}$ yields a revision rule, $\delta_{\mathscr{D},M} : \wp D \rightarrow \wp D$, thus: for all hypotheses h and all $d \in D$,

$$d \in \delta_{\mathscr{D},M}(h) \text{ iff } d \text{ satisfies } A(x, G) \text{ in } M + h.$$

So, given a hypothetical antecedent extension h for the defined term G, the revision rule yields a revised extension for G. This revised extension consists of members of D that satisfy the definiens under the assumption that G is assigned the interpretation h.[6]

A revision rule $\rho : \wp D \rightarrow \wp D$ may be applied repeatedly to an initial hypothesis. Let us define the notion "the result of n applications of ρ to a hypothesis h" (notation: $\rho^n(h)$) recursively:

$\rho^0(h) = h$, and

$\rho^{n+1}(h) = \rho(\rho^n(h))$.[7]

We wish to isolate hypotheses, to be called *saturated* hypotheses, that result when the applications of the revision rule are iterated to the point of saturation. One set of such hypotheses are those that are *descending*, where a hypothesis h is *descending for* ρ iff there exist hypotheses h_n, $n \geq 0$, such that

(i) $h_0 = h$, and

(ii) $h_n = \rho(h_{n+1})$.

A descending hypothesis is a result of infinitely many applications of the revision rule. There is another way of gaining saturated sets, and it involves a different way of applying the revision rule infinitely many times. This way is found in *revision sequences*, to which let us now turn.

Let On be the class of all ordinals. Let $\mathscr{S}$ be an On-long sequence of subsets of D, and let $\mathscr{S}_\beta$ be the β^{th} member of $\mathscr{S}$. If α is a limit ordinal, then we say of an element $d \in D$ that it is *stably in* [*stably out of*] $\mathscr{S}$ *at* α iff

$$\exists\gamma < \alpha \forall\beta (\text{if } \gamma \leq \beta < \alpha \text{ then } d \in [\notin]\ \mathscr{S}_\beta).$$

Let us say that d is *stable in* $\mathscr{S}$ *at* α iff d is either stably in or stably out of $\mathscr{S}$ at α; otherwise, let us say that d is *unstable in* $\mathscr{S}$ *at* α. A hypothesis h is said to *cohere with* $\mathscr{S}$ *at* a limit ordinal α iff

(i) if an element $d \in D$ is stably in $\mathscr{S}$ at α then $d \in h$, and

(ii) if an element $d \in D$ is stably out of $\mathscr{S}$ at α then $d \notin h$.

Finally, $\mathscr{S}$ is a *revision sequence for* ρ iff $\mathscr{S}$ is an On-long sequence of hypotheses such that, for all ordinals α and β,

(i) if $\alpha = \beta + 1$ then $\mathscr{S}_\alpha = \rho(\mathscr{S}_\beta)$, and

(ii) if α is limit then $\mathscr{S}_\alpha$ coheres with $\mathscr{S}$ at α.[8]

It is easy to show that all revision sequences $\mathscr{S}$ contain *cofinal* hypotheses. That is, hypotheses h exist that satisfy the following condition:

$$\forall\alpha\exists\beta \geq \alpha(\mathscr{S}_\beta = h).$$

We say that a hypothesis h is *recurring for* a revision rule ρ iff h is cofinal in a revision sequence for ρ.[9]

We can now define the crucial notion on which scheme **S** is built: a hypothesis h is *saturated for* ρ iff h is either recurring or descending for ρ.[10] The notion "in **S**, M $\mathscr{D}$-validates B" ($M \overset{\mathscr{D}}{\models} B$) now receives the following definition:

(3) $M \overset{\mathscr{D}}{\models} B$ iff $\exists p\forall h$(if h is saturated for $\delta_{\mathscr{D},M}$ then $\forall n \geq p[B$ is true in $M + \delta^n_{\mathscr{D},M}(h)]$).

Other semantical notions such as "a sentence being paradoxical in M" can also be defined. For the sake of brevity, we pass over these notions and restrict ourselves to a reflection on "validity" and "implication." The latter notion can be recovered from "$M \overset{\mathscr{D}}{\models} B$" in the way indicated above, through (1) and (2). Below we shall compare **S** with other semantical schemes. In characterizing these schemes, we shall define, as with **S**, the distinctive notion of "sentence B being valid in a ground model M" underlying each scheme. This will suffice to fix, through the analogs of (1) and (2), the notion of implication associated with each scheme.

Let us take note of the form of the definition of validity, (3), in **S**:

$M \overset{\mathscr{D}}{\models} B$ iff $\exists p\forall h$(if h is ______ for $\delta_{\mathscr{D},M}$ then $\forall n \geq p[B$ is true in $M + \delta^n_{\mathscr{D},M}(h)]$).

Let us call a definition of validity of this form (as well as the scheme to which the definition belongs) "Type 1" or, the mnemonically easier, "$\exists p\forall h$-type." We shall see that this form, though complex, is in part responsible for the special features of **S**. We lose some of the features if we opt for a definiens of the following simpler form:

$$\forall h(\text{if } h \text{ is } \_\_\_\_\_ \text{ for } \delta_{\mathscr{D},M} \text{ then } B \text{ is true in } M + h).$$

Below we shall look at some schemes in which the definition of validity uses a definiens of this form. Let us say that such schemes (and their definitions of validity) are of "Type 2" or "$\forall h$-type." There will be occasion to mention schemes of a third type in which the definition of validity is formulated using a definiens similar to that in Type 1 except that the two initial quantifiers are switched:

$$\forall h \exists p(\text{if } h \text{ is } \_\_\_\_\_ \text{ for } \delta_{\mathscr{D},M} \text{ then } \forall n \geq p[B \text{ is true in } M + \delta^n_{\mathscr{D},M}(h)]).$$

Let us say that such schemes (and their definitions of validity) are of "Type 3" or "$\forall h \exists p$-type." These schemes are less well understood than those of the first two types.

2 Some Examples

2.1 Example. Let $\mathscr{L}$ contain names a and b and a one-place predicate H. Let the domain of model M be the set of natural numbers $\mathbb{N}$ and let the interpretation function I assign to the names a and b, respectively, 0 and 1, and let I assign to the predicate H the set of prime numbers P. Finally, let the definiens of $\mathscr{D}$ be

$$Hx \vee [(x = a \vee x = b) \;\&\; \neg Ga \;\&\; \neg Gb].$$

Then the revision $\delta_{\mathscr{D},M}$ is as follows: for all $h \subseteq \mathbb{N}$,

$$\delta_{\mathscr{D},M}(h) = \begin{cases} P \cup \{0,1\} & \text{if } 0 \notin h \text{ and } 1 \notin h; \\ P, & \text{otherwise.} \end{cases}$$

The descending hypotheses of this revision rule are P and $P \cup \{0,1\}$, and its recurring hypotheses are:

$$P, P \cup \{0\}, P \cup \{1\}, \text{ and } P \cup \{0,1\}.$$

It can be verified that neither Ga nor $\neg Ga$ is $\mathscr{D}$-valid in M but that the following two sentences are $\mathscr{D}$-valid in M:

$$(\neg Ha \;\&\; \neg Hb) \supset (Ga \equiv Gb) \text{ and } \forall x(Gx \supset x = a \vee x = b \vee Hx).$$

These sentences are also absolutely $\mathscr{D}$-valid (i.e., $\mathscr{D}$-valid in all ground models) and thus $\mathscr{D}$-implied in $\mathbf{S}$ by $\emptyset$. On the other hand, a sentence of $\mathscr{L}^+$ that says "there are either two Gs that are not prime or there are no such Gs" is $\mathscr{D}$-valid in M, but it is not absolutely $\mathscr{D}$-valid and thus not $\mathscr{D}$-implied by $\emptyset$.

Remarks. (i) The above example shows that a concept with a circular definition does not, in general, sharply carve the domain into objects that fall under the concept and those that do not. In the above example, the status of 0 and 1 fluctuates through the revision process; in contrast, the status of all other objects is fully settled. Let us say that an object $d \in D$ *categorically falls under* [*categorically fails to fall under*] G *in* $\mathbf{S}$ *relative to* $\mathscr{D}$ *and* M iff

$$\exists p \forall h (\text{if } h \text{ is saturated for } \delta_{\mathscr{D},M} \text{ then } \forall n \geq p (d \in \delta^n_{\mathscr{D},M}(h)\ [d \notin \delta^n_{\mathscr{D},M}(h)])).$$

The categorical range of G *in* $\mathbf{S}$ *relative to* $\mathscr{D}$ *and* M, let us stipulate, is the set of those objects that either categorically fall under G or categorically fail to fall under G in $\mathbf{S}$ relative to $\mathscr{D}$ and M. In the above example, all and only prime numbers categorically fall under G, and the categorical range of G is $\mathbb{N} - \{0, 1\}$. Note that if c is a name of an object d in the domain then:

> d categorically falls under G [categorically fails to fall under G] in $\mathbf{S}$ relative to $\mathscr{D}$ and M iff $M \models^{\mathscr{D}} Gc$ [$M \models^{\mathscr{D}} \neg Gc$].

This connection between categoricalness and validity holds under the other semantical schemes considered below.

(ii) The above example shows that when G is defined circularly, "the extension of G" does not always exist. However, sometimes, as in this example, the defined term can be viewed as possessing what may be called a *superextension.* That is, "validity relative to M" sometimes boils down to truth relative to all hypotheses in a particular set, the superextension of the defined term. So, if $\mathscr{X}$ is the superextension, we have:

$$M \models^{\mathscr{D}} B \text{ iff } \forall h (\text{if } h \in \mathscr{X} \text{ then } B \text{ is true in } M + h).$$

Note that the superextension in $\mathbf{S}$, if it exists, is necessarily a subset of the set of saturated hypotheses, but it is not necessarily identical to this set. In the above example, the superextension of G is $\{P, P \cup \{0, 1\}\}$, and the saturated hypotheses $P \cup \{0\}$ and $P \cup \{1\}$ do not belong to it.

2.2 Example. Let $\mathscr{L}$ contain the two-place predicate R. Let the domain of the ground model M be a limit ordinal α $(= \{\beta : \beta < \alpha\})$, and let the interpretation I assign to R the relation $<$ restricted to α. Let the definiens of $\mathscr{D}$ be

$$\forall y(Ryx \supset Gy).$$

Now the revision rule $\delta_{\mathscr{D},M}$ is as follows: for all hypotheses h,

$$\delta_{\mathscr{D},M}(h) = \begin{cases} \alpha & \text{if } h = \alpha; \\ \gamma \cup \{\gamma\} & \text{if } h \neq \alpha \text{ and } \gamma \text{ is the least ordinal not in } h. \end{cases}$$

Every revision sequence $\mathscr{S}$ of $\delta_{\mathscr{D},M}$ culminates in α. That is, α is cofinal in $\mathscr{S}$, and it is the only hypothesis cofinal in $\mathscr{S}$.[11] Hence, α is the only hypothesis that is recurring for $\delta_{\mathscr{D},M}$. It can be verified that it is also the only hypothesis that is descending for $\delta_{\mathscr{D},M}$. So, there is a unique saturated hypothesis for $\delta_{\mathscr{D},M}$, and "$\mathscr{D}$-validity in **S** relative to M" reduces to "truth in $M + \alpha$."[12] It follows that, for every sentence B of $\mathscr{L}^+$, either B or $\neg B$ is $\mathscr{D}$-valid in M; in particular, $\forall xGx$ is $\mathscr{D}$-valid in M. This sentence is not, however, absolutely $\mathscr{D}$-valid. Indeed, the following stronger claim holds:

(4) $\emptyset \not\models^{\mathscr{D}} \forall x[\forall y(Ryx \supset Gy) \supset Gx]$.

For observe, first, this general fact: for any ground model M^* and any descending hypothesis h, if a sentence B is false in $M^* + h$ then B is not valid relative to M^* and, hence, not absolutely valid. Now, let M^* be a ground model whose domain is the set of all integers and in which R is interpreted as the relation $<$ over the integers. It is easily seen that, if j is an arbitrary integer, then the hypothesis $h = \{i : i < j\}$ is descending. But $\forall x[\forall y(Ryx \supset Gy) \supset Gx]$ is false in $M^* + h$. It follows that (4) holds.

Remarks. (i) This example illustrates that sometimes a circularly defined predicate is fully classical: the categorical range of the predicate is identical to the domain. In such cases, and only in such cases, it is legitimate to speak of *the* extension of the predicate. And, in such cases, the extension of the predicate is the unique saturated hypotheses h, and the superextension is the set $\{h\}$.

(ii) The example shows also the importance of not putting any bound on lengths of revision sequences. If revision sequences were required to be of length less than α, then in the above example the interpretation of G in a model with the domain $\alpha + \omega$ would fail to be classical. Sometimes we reach "good" hypotheses for the interpretation of the defined term only after transfinitely many "bad" ones have been thrown away in the course of revision.

2.3 Example. Let $\mathscr{L}$ contain the name 0, the one-place function symbol $'$, and the two-place predicate R. Let us understand $\overline{n}$ in the usual way: $\overline{n}$ is the term consisting of n applications of $'$ to 0. Let us assume, as in the previous example, that the domain of the ground model M is a limit ordinal α and that R is interpreted as $<$ restricted to α. Let us stipulate that the interpretation of the name 0 is the ordinal 0 and that the interpretation of the function symbol $'$ is the successor function restricted to α. Let 'G *is closed*' abbreviate

$$G(0) \ \& \ \forall x(G(x) \supset G('(x))).$$

Finally, set the definiens of $\mathscr{D}$ to be

$$[G \textit{ is closed} \ \& \ x \neq x] \vee [\neg(G \textit{ is closed}) \ \& \ \forall y(Ryx \supset Gy)].$$ [13]

Let us say that a hypothesis h is *closed* iff h contains 0 as well as the successor of any ordinal that belongs to h. Then the revision rule $\delta_{\mathscr{D},M}$ is as follows: for all hypotheses h,

$$\delta_{\mathscr{D},M}(h) = \begin{cases} \emptyset & \text{if } h \text{ is closed;} \\ \gamma \cup \{\gamma\} & \text{if } h \text{ is not closed and } \gamma \text{ is the least ordinal not in } h. \end{cases}$$

This revision rule, unlike the previous two, yields no descending hypotheses. Furthermore, all and only subsets of ω are recurring for $\delta_{\mathscr{D},M}$.[14] The following sentences, it can be verified, are $\mathscr{D}$-valid in M in **S**:

$$\exists x \neg G(x), \ G(0), \ G('(0)), \ G('('(0))), \ \ldots, \ G(\overline{n}), \ \ldots.$$

They are not, however, absolutely $\mathscr{D}$-valid.

Remark. This example shows that sometimes neither an extension nor a superextension can be associated with a circularly defined predicate. If in the above example we set the domain of the ground model M to be ω then no superextension captures the set of sentences validated by M. Possession of an extension or even a superextension is not a prerequisite for a predicate to be meaningful. Sometimes the semantical behavior of a predicate can be captured only through an irreducible revision process, not through an assignment to the predicate of an extension or a superextension.

3 Features of S

I wish to draw attention to five features of **S**.

(i) *Conservativeness*. **S** is *conservative*, in the sense that, for all definitions $\mathscr{D}$ and all ground models M and all sentences B of ground language $\mathscr{L}$, if B is $\mathscr{D}$-valid in **S** relative to M then B is true in M. (This holds because saturated hypotheses are bound to exist for revision rules.) So, under scheme **S**, the addition of a definition does not disturb the truth and falsity of sentences in the ground language; nor does it disturb the interpretation of the vocabulary in the ground language. **S** leaves the ground language as is. The changes it institutes concern only the new vocabulary introduced by the added definitions.

It is easy to construct semantical schemes that violate conservativeness. Consider a Type-1 scheme that is based on descending hypotheses (as opposed to both descending and recurring hypotheses invoked in **S**). Let us call this scheme $\mathbf{S}_{1d}$. Here the subscript '1' indicates that the scheme is of Type 1 and 'd' indicates that the scheme is based on descending hypotheses. (Under this nomenclature, $\mathbf{S} = \mathbf{S}_{1s}$, where the marker 's' indicates that the scheme is based on saturated hypotheses.) Now, if definition $\mathscr{D}$ and ground model M are as specified in Example 2.3, then no hypotheses are descending for the revision rule. Consequently, $\perp$ is $\mathscr{D}$-valid in $\mathbf{S}_{1d}$ relative to M, and conservativeness fails.

(ii) *Preservation of Ground Logic*. **S** preserves ground logic, in the sense that the logic of the ground language carries over to the extended language: forms of argument that were logically valid before the introduction of the definitions remain valid after their introduction. Hence, in **S**, one reasons with the new vocabulary as one reasons with the old vocabulary. If, for example, *modus ponens* was a valid form of argument in the ground language then it remains a valid form in the extended language. **S** thus respects the idea that definitions do not bestow new meanings on the logical constants in the ground language. Definitions bestow meanings only on the defined terms, and they do so by exploiting the preexisting meanings of ground logical constants; they do not alter these preexisting meanings.

In introducing **S** above, I took the ground language to be classical. It is worth noting, though, that the idea underlying **S** is general. Circular and interdependent definitions can be added to non-classical languages also,[15] and scheme **S** carries over naturally to yield a semantic account of them. In each case, **S** respects the ground logic. If the ground logic is intuitionistic, for example, then so also is the logic of the extended language.

(iii) *Rich Content.* **S** attributes a rich content to the defined terms. Consider, for comparison, the Type-1 scheme, $\mathbf{S}_{1a}$, that is based on *all* hypotheses.[16] Under this scheme, the content attributed to the defined terms can be quite weak. If $\mathscr{D}$ and M are as specified in Example 2.2 and if we set the domain of M to be $\omega + \omega$, then the categorical range of G under $\mathbf{S}_{1a}$ is only ω. Under **S**, in contrast, it is the whole domain, $\omega + \omega$. As noted in Remark (ii) following the example, **S** attributes rich content to defined terms because it imposes no bound on the number of times the revision rule may be applied in the course of revision; the revision rule may be iterated to the point of saturation. Under $\mathbf{S}_{1a}$, however, the iterations of the revision rule are highly constrained.

(iv) *Preservation of Finite Natural Implication.* There is a natural implication relation over a class of definitions, called *finite* definitions, and **S** preserves this relation. Finite definitions can be characterized as follows. Let us say that a hypothesis h of a revision rule ρ is *finitely reflexive* iff, for some $n > 0$, $\rho^n(h) = h$. Now, a definition $\mathscr{D}$ is *finite* iff for all ground models M there is number n such that, for all hypothesis h (relative to M):

$\delta^n_{\mathscr{D},M}(h)$ is finitely reflexive.

If we think of the successive applications of a revision rule to arbitrary hypotheses as generating a revision process that filters out "bad" hypotheses then, as we have seen, this process can require transfinitely many stages before it begins to yield only "good" hypotheses. The distinctive feature of finite definitions is that their revision processes filter out all the "bad" hypotheses in the course of finitely many stages. Irrespective of the ground model, the revision rule yields, after a certain finite number of iterations, only "good" hypotheses (= finitely reflexive hypotheses). Note that the definition considered in Example 2.1 is finite, and those considered in Examples 2.2 and 2.3 are not finite.

Finite definitions can be given a simple semantics, which, in turn, yields a natural implication relation. Let $\mathbf{S}_{2f}$ be the Type-2 scheme based on finitely reflexive hypotheses. So, the notion of validity (notation: $M \models^{\mathscr{D}}_{2f} B$) is defined as follows under this scheme:

$$M \models^{\mathscr{D}}_{2f} B \text{ iff } \forall h (\text{if } h \text{ is finitely reflexive for } \delta_{\mathscr{D},M} \text{ then } B \text{ is true in } M + h).$$

What I am calling "finite natural implication" is the notion of implication ("$A_1, \ldots, A_n \models^{\mathscr{D}}_{2f} C$") that "validity in $\mathbf{S}_{2f}$" yields over finite definitions. Now, it is easily verified that if $\mathscr{D}$ is finite then

$$A_1, \ldots, A_n \models^{\mathscr{D}} C \text{ iff } A_1, \ldots, A_n \models^{\mathscr{D}}_{2f} C.$$

Indeed, for all models M and finite definitions $\mathscr{D}$ and sentences B of $\mathscr{L}^+$,

$$M \models^{\mathscr{D}} B \text{ iff } M \models^{\mathscr{D}}_{2f} B.$$

So, schemes $\mathbf{S}$ and $\mathbf{S}_{2f}$ coincide over finite definitions. Note that $\mathbf{S}_{2f}$ is not conservative; so, it does not itself provide a satisfactory general logic of definitions.[17]

(v) *Axiomatizability.* The notion "implication in $\mathbf{S}$" is axiomatizable. I provide in §5 a logical calculus that is sound and complete with respect to $\mathbf{S}$.

4 Comparisons

If we take, as I think we should, Conservativeness, Preservation of Ground Logic, and Rich Content as non-negotiable desiderata of a general logic of definitions then three schemes stand out as competitors of $\mathbf{S}$.[18] These are: $\mathbf{S}^\#$, $\mathbf{S}^*$, and $\mathbf{S}^\sigma$. Of these, $\mathbf{S}^\#$ is a Type-1 scheme and is based on recurring hypotheses. Letting 'r' be the marker for recurring hypotheses, we have $\mathbf{S}^\# = \mathbf{S}_{1r}$. So:

($\mathbf{S}^\#$) A sentence B *is $\mathscr{D}$-valid in* $\mathbf{S}^\#$ *relative to* model M (notation: $M \models^{\mathscr{D}}_{\#} B$) iff $\exists p \forall h$(if h is recurring for $\delta_{\mathscr{D},M}$ then $\forall n \geq p[B$ is true in $M + \delta^n_{\mathscr{D},M}(h)]$).

Scheme $\mathbf{S}^*$ ($= \mathbf{S}_{2r}$) is also based on recurring hypotheses, but it is of Type 2:

($\mathbf{S}^*$) A sentence B *is $\mathscr{D}$-valid in* $\mathbf{S}^*$ *relative to* model M (notation: $M \models^{\mathscr{D}}_{*} B$) iff $\forall h$(if h is recurring for $\delta_{\mathscr{D},M}$ then B is true in $M + h$).

Finally, $\mathbf{S}^\sigma$, too, is a Type-2 scheme, but it is based on "σ-acceptable" hypotheses:[19]

($\mathbf{S}^\sigma$) A sentence B *is $\mathscr{D}$-valid in* $\mathbf{S}^\sigma$ *relative to* model M (notation: $M \models^{\mathscr{D}}_{\sigma} B$) iff $\forall h$(if h is σ-acceptable for $\delta_{\mathscr{D},M}$ then B is true in $M + h$).

The notion of "σ-acceptable" hypothesis is defined as follows. Let $\rho : \wp D \to \wp D$ be an arbitrary revision rule. Let $\overline{\rho} : \wp\wp D \to \wp\wp D$ be the operation such that, for all $\mathscr{X} \subseteq \wp D$ and all $Z \subseteq D$, $Z \in \overline{\rho}(\mathscr{X})$ iff

(i) $\bigcap\{\rho(Y) \mid Y \in \mathscr{X}\} \subseteq Z$, and

(ii) $\bigcap\{D - \rho(Y) \mid Y \in \mathscr{X}\} \subseteq D - Z$.

Operation $\overline{\rho}$ is *monotone* on $\wp\wp D$, in the sense that, for all $\mathscr{X}, \mathscr{Y} \subseteq \wp D$, if $\mathscr{X} \subseteq \mathscr{Y}$ then $\overline{\rho}(\mathscr{X}) \subseteq \overline{\rho}(\mathscr{Y})$. Hence, by the Knaster-Tarski theorem, $\overline{\rho}$ has a largest fixed point. Now, a hypothesis is *σ-acceptable for ρ* iff it belongs in this fixed point.

It can be verified that all saturated hypotheses are σ-acceptable. It follows that $\mathbf{S}^\sigma$ is weaker than $\mathbf{S}$ (notation: $\mathbf{S}^\sigma \leq \mathbf{S}$), in the sense that

> for all definitions $\mathscr{D}$ and ground models M and sentences B of $\mathscr{L}^+$, if $M \models^{\mathscr{D}}_{\sigma} B$ then $M \models^{\mathscr{D}} B$.

Indeed, $\mathbf{S}^\sigma$ is the weakest of the four systems, and $\mathbf{S}^\#$ is the strongest. $\mathbf{S}$ and $\mathbf{S}^*$, on the other hand, are incomparable to one another. The relationships between the systems may therefore be pictured thus:

$$\begin{array}{ccccc} & & \mathbf{S}^* & & \\ & \leq & & \leq & \\ \mathbf{S}^\sigma & & & & \mathbf{S}^\# \\ & \leq & & \leq & \\ & & \mathbf{S} & & \end{array}$$

Of these, the two $\forall h$-type schemes, namely $\mathbf{S}^\sigma$ and $\mathbf{S}^*$, do not preserve finite natural implication. The following implication,

$$\emptyset \models^{\mathscr{D}} (\neg Ha \,\&\, \neg Hb) \supset (Ga \equiv Gb),$$

holds in $\mathbf{S}$ (and hence also in $\mathbf{S}_{2f}$) for the finite definition $\mathscr{D}$ considered in Example 2.1. But it fails in $\mathbf{S}^*$ (and hence also in $\mathbf{S}^\sigma$).

Of the remaining two schemes, $\mathbf{S}^\#$ is not axiomatizable. This was established by Philip Kremer.[20] Kremer showed that the complexity of the implication relation of $\mathbf{S}^\#$ is high indeed: it is $\mathbf{\Pi}^1_2$.[21] This result holds also for $\mathbf{S}^*$. I do not know where things stand with respect to $\mathbf{S}^\sigma$:

4.1 Problem. Is the implication relation of $\mathbf{S}^\sigma$ axiomatizable? If not, what is its complexity?

So, among the principal schemes, $\mathbf{S}$ is the only one that possesses the above five features: (i) Conservativeness, (ii) Preservation of Ground Logic, (iii) Rich Content, (iv) Preservation of Finite Natural Implication, and (v) Axiomatizability. It may be possible to strengthen $\mathbf{S}$—say by restricting oneself to a subclass of saturated hypotheses or by moving to an $\forall h \exists p$-type definition of validity—and still preserve these five features. Whether this is a genuine possibility is an open question. A more specific open question here is this:

4.2 Problem. Is the implication relation of scheme $\mathbf{S}_{3s}$ axiomatizable? If not, what is its complexity?

I note that a similar question is open for $\mathbf{S}_{3a}$, though of course this scheme does not provide a satisfactory account of circular definitions:

4.3 Problem. Is the implication relation of scheme $\mathbf{S}_{3a}$ axiomatizable? If not, what is its complexity?

So, to repeat, a *strengthening* of $\mathbf{S}$ may be possible that preserves the five features. However, as I argue in the next section, the prospects are dim for preserving these features while *weakening* **S**. I argue that **S** yields the minimal general logic of definitions.

5 A Logic of Definitions

The notion of implication in **S** admits of a strikingly simple axiomatization. The calculus $\mathbf{C_0}$ ($=_{Df}$ **C**) of *RTT* is sound and complete with respect to **S**. In this calculus, one reasons with *indexed formulas*, B^i, which are pairs consisting of formulas of $\mathscr{L}^+$ (B) and integer indices (i). Intuitively, one may think of an indexed formula B^i as saying that the formula B holds, relative to a certain assignment of values to the variables, at a revision stage labeled i of an arbitrary revision sequence. To make room for the possibility that some stages may be labeled with negative integers, let us allow that a revision stage high in the sequence may be labeled 0. Let us use capital Greek letter Γ to range over sets of indexed formulas. And let us abbreviate "relative to $\mathscr{D}$, B^j is derivable in **C** from Γ" (alternatively, "B^j is $\mathscr{D}$-derivable in **C** from Γ") thus:

$$\Gamma \vdash^{\mathscr{D}} B^j,$$

where let us understand definition $\mathscr{D}$ to be, as before,

$$Gx =_{Df} A(x, G).$$

Then the following four rules govern reasoning with indexed formulas in **C**:

(i) *Classical Logic* (CL). If a formula B is a classical consequence of formulas $A_1, \ldots, A_n$ and there is an index i such that, for all $1 \leq m \leq n$, $\Gamma \vdash^{\mathscr{D}} A^i_m$ then $\Gamma \vdash^{\mathscr{D}} B^i$. This rule reflects the fact that the ground logic is classical logic and the ground logic is preserved at each revision stage.

(ii) *Index Shift* (IS). If B does not contain any occurrences of any defined term and $\Gamma \vdash^{\mathscr{D}} B^i$ then, for all integers j, $\Gamma \vdash^{\mathscr{D}} B^j$. This rule reflects the fact that the semantic status of a ground-language formula does not change across revision stages. For example, if a ground-language sentence is true at one stage, then it is true at all stages.

(iii) *Definiendum Introduction* ($\mathrm{DfI_r}$).[22] If $\Gamma \vdash^{\mathscr{D}} A(t, G)^i$, where t is free for x in $A(x, G)$, then $\Gamma \vdash^{\mathscr{D}} Gt^{i+1}$. This rule reflects the fact that if the definiens holds at one revision stage, then the definiendum holds at the next revision stage.

(iv) *Definiendum Elimination* ($\mathrm{DfE_r}$). If $\Gamma \vdash^{\mathscr{D}} Gt^{i+1}$ and t is free for x in $A(x, G)$ then $\Gamma \vdash^{\mathscr{D}} A(t, G)^i$. This rule reflects the converse fact, namely, that if the definiendum holds at the revision stage $i + 1$, then the definiens holds at stage i. Note that indices do real work only when definitions are in play and only when the definitions create certain kinds of dependencies.[23]

Derivability of formulas and indexed formulas are linked by the following principle:

$$A_1 \ldots A_n \vdash^{\mathscr{D}} B \text{ iff } A_1^0 \ldots A_n^0 \vdash^{\mathscr{D}} B^0.$$

For an example of a derivation, recall the definition $\mathscr{D}$ considered in Example 2.1:

$$Gx =_{Df} Hx \vee [(x = a \vee x = b) \;\&\; \neg Ga \;\&\; \neg Gb].$$

The following derivation verifies that $\neg Ha,\ \neg Hb,\ Ga \vdash^{\mathscr{D}} Gb$:

(i)	$\neg Ha^0$ (ii) $\neg Hb^0$ (iii) Ga^0	(Assumptions)
(iv)	$(Ha \vee [(a = a \vee a = b) \;\&\; \neg Ga \;\&\; \neg Gb])^{-1}$	(iii, $\mathrm{DfE_r}$)
(v)	$\neg Ha^{-1}$	(i, IS)
(vi)	$(\neg Ga \;\&\; \neg Gb)^{-1}$	(iv & v, CL)
(vii)	$(Hb \vee [(b = a \vee b = b) \;\&\; \neg Ga \;\&\; \neg Gb])^{-1}$	(vi, CL)
(viii)	Gb^0	(vii, $\mathrm{DfI_r}$)

Calculus **C** uses indices that are external to the language to keep track of revision stages. This tracking can be accomplished also by a device internal to the language. We can view revision stages as possible situations, and we can add to the language a modal operator $\Box$ that has the force of "it is true in the previous revision stage that ... ." The details of this can be found in Shawn Standefer's and my paper "Conditionals in Theories of Truth."[24] The point I wish to note here is that the rules of inference governing definition $\mathscr{D}$ are now captured by the principle:

(Df$\Box$) $\forall x(Gx \equiv \Box A(x, G))$.

If the definition is non-circular, then this principle implies (through an analog of Index Shift) that

$$\forall x(Gx \equiv A(x, G)).$$

In general, though, this implication does not hold (and cannot hold, for otherwise conservativeness would be violated).

Calculus **C**, I wish to argue, is weak. Furthermore, prospects are dim for working up a viable general logic of definitions that weakens **C**. More specifically, the rules of inference $\mathrm{DfI_r}$ and $\mathrm{DfE_r}$—or some analog of them, such as $\mathrm{Df}\Box$—must be countenanced by any viable general logic of definitions. Let me offer four considerations in support of these claims.

(i) **C** is sound with respect to $\mathbf{S}_{1a}$.[25] Hence, **C** is sound with respect to *every* Type-1 and Type-3 scheme. If Type-2 schemes are ruled out then **C** is the minimal general logic for definitions.

(ii) **C**, or some calculus equivalent to it, is needed, it appears, to preserve finite natural implication. Every known Type-2 scheme fails to sustain finite natural implication and fails also to sustain **C**. Preservation of finite natural implication is, it seems to me, a requirement on a general logic of definitions. Not only are such implications intuitively compelling. They are needed in applications of the logic—for example, to the concept of rational choice.[26]

(iii) It will not do to go for a disjunctive Type-2 scheme, one that interprets finite definitions using finitely reflexive hypotheses and interprets the non-finite definitions in a different way. This would yield two different sets of logical rules for reasoning with definitions, one set for finite definitions and the other for non-finite ones. This is unacceptable, for finite definitions do not constitute a recursive set.[27] The rules for working with the two sorts of definitions should be uniform. And there can be no doubt that the rules **C** prescribes for working with finite definitions are correct.

(iv) The heart of **C** are the rules $\mathrm{DfI_r}$ and $\mathrm{DfE_r}$, and these rules capture the core semantic idea underlying all revision theories: using the interpretation of the definiens at one stage to move to the interpretation of the definiendum at the next stage. The violation of these rules by Type-2 schemes should be seen as a defect in these schemes—a defect that arises because these schemes give improper weight to certain hypotheses (for example, the hypotheses $P \cup \{0\}$ and $P \cup \{1\}$ in Example 2.1).

If **C** is indeed the minimal general logic of definitions then every general logic must tolerate ω-inconsistency. For, relative to some definitions, ω-inconsistent sets of sentences can be deduced in **C** from weak arithmetical principles. An example is provided by definition $\mathscr{D}$ given in Example 2.3:

$$Gx =_{Df} [G\ is\ closed\ \&\ x \neq x] \vee [\neg(G\ is\ closed)\ \&\ \forall y(Ryx \supset Gy)],$$

where, recall, 'G *is closed*' abbreviates

$$G(0)\ \&\ \forall x(G(x) \supset G('(x))).$$

Let AX be the conjunction of axioms stating that R is a strict linear ordering and, furthermore, that R meets the following two conditions:

(i) $\forall x \neg Rx0$,

(ii) $\forall x(R(x, '(x))\ \&\ \forall y(Rxy \supset y = '(x) \vee R('(x), y))$.

Then it can be verified that

$$AX \vDash^{\mathscr{D}} G\overline{n}, \text{ for all } n \geq 0, \text{ and}$$

$$AX \vDash^{\mathscr{D}} \exists x \neg Gx.$$

So, a minimal logic allows the deduction of an ω-inconsistency from weak principles. I have come to think that we should accept this for what it is: an interesting fact but not one that is a threat to the minimal logic of definitions. For note, first, that the logic does not *force* us to accept definitions that generate ω-inconsistencies. All it does is to provide us with an *option* to accept such definitions if we find it useful to do so. Furthermore, if we do accept one of these definitions, the logic does not force a non-standard interpretation on our ordinary arithmetical vocabulary. The acceptance of the definition entails no changes at all in the interpretation of the ground vocabulary. The acceptance entails merely that the semantic behavior of the newly defined term can be captured only through an irreducible revision process. The behavior cannot be captured through an assignment of, for example, a superextension to the term.

6 Application to Truth

If, following Alfred Tarski, we take the biconditionals of the form,

(5) $T(\text{`}A\text{'})$ iff A,

to be partial definitions of the truth-predicate T, then T expresses a circular concept and we can apply the different semantical schemes for circular definitions to obtain different theories of truth.[28] The theory of truth, **T**, that scheme **S** yields is, I wish to point out, attractive, and furthermore, some of its attractiveness is tied essentially to **S**'s tolerance of ω-inconsistency. Let me draw attention to three attractive features of **T**. The first of these is independent of **S**'s tolerance of ω-inconsistency; the remaining two features are not.

(i) **T** attributes a rich content to the truth-concept. In particular, sentences grounded true in a model M under the Strong Kleene version of Kripke's theory of truth are assessed as valid in M under **T**. This feature, I should note, is possessed also by theories of truth $\mathbf{T}^*$ and $\mathbf{T}^\sigma$, based respectively on schemes $\mathbf{S}^*$ and $\mathbf{S}^\sigma$, even though these two schemes do not tolerate ω-inconsistency.

(ii) **T** interprets the biconditionals (5) in a way that sustains natural ways of reasoning with truth. The logical force of (5) is captured, under **T**, by the principle

$$T(\text{`}A\text{'}) \equiv \Box A,$$

where $\Box$ is interpreted as indicated above. Equivalently, in **T**, the truth-concept is governed by the following rules of inference:

Truth Introduction ($\mathrm{TI_r}$) A^i; therefore $T(\text{`}A\text{'})^{i+1}$.[29]

Truth Elimination ($\mathrm{TE_r}$) $T(\text{`}A\text{'})^{i+1}$; therefore A^i.

Here is a simple example of reasoning that these rules help sustain:

(i)	$T(`A \supset B\text{'})^0$	(Assumption)
(ii)	$T(`A\text{'})^0$	(Assumption)
(iii)	$(A \supset B)^{-1}$	(i, $\mathrm{TE_r}$)
(iv)	A^{-1}	(ii, $\mathrm{TE_r}$)
(v)	B^{-1}	(iii, iv, CL)
(vi)	$T(`B\text{'})^0$	(v, $\mathrm{TI_r}$)
(vii)	$(T(`A\text{'}) \supset T(`B\text{'}))^0$	(ii-vi, CL)
(viii)	$[T(`A \supset B\text{'}) \supset (T(`A\text{'}) \supset T(`B\text{'}))]^0$	(i-vii, CL)

So, if the conditional $(A \supset B)$ is true, then the consequent B is true if the antecedent A is true. A parallel argument establishes the converse. Hence, **T** interprets the rules for truth in a way that sustains the natural reasoning to the conclusion that

$$T(`A \supset B\text{'}) \equiv (T(`A\text{'}) \supset T(`B\text{'})).$$

This reasoning, I wish to point out, is sustained neither in $\mathbf{T}^\sigma$ nor in $\mathbf{T}^*$. And this is not an isolated phenomenon: examples like this can be multiplied indefinitely. To sustain simple kinds of reasoning with the truth-concept, we need rules of inference $\mathrm{TI_r}$ and $\mathrm{TE_r}$ (or their analogues), and this requires a scheme that validates $\mathrm{DfI_r}$ and $\mathrm{DfE_r}$. We have seen that such a scheme will tolerate ω-inconsistency.

(iii) **T** allows us to affirm semantical laws. Assume that the language has the following predicates

$Conj(x, y, z)$:	z is a sentence that is a conjunction of x and y;
$Neg(x, z)$:	z is a sentence that is a negation of x;
$Univ(x, z)$:	z is a universal sentence and sentence x is an instance of it;
$Den(x, y)$:	x is a term that denotes y.

Assume, furthermore, that these predicates receive the intended interpretation in the ground model M. Then, under **T**, the following semantical laws are assessed as valid in M:

(i) $\forall x, y, z[Conj(x, y, z) \supset (Tz \equiv Tx \;\&\; Ty)]$;

(ii) $\forall x, z[Neg(x, z) \supset (Tz \equiv \neg Tx)]$; and

(iii) $\forall z[\exists x Univ(x, z) \;\&\; \forall y \exists x Den(x, y) \supset (Tz \equiv \forall x(Univ(x, z) \supset Tx)]$.

These semantical laws are not only natural and intuitive; they are also useful in reasoning with the concept of truth.[30] But these laws can be sustained only in theories of truth that are based on schemes that tolerate ω-inconsistency. For, as Vann McGee has shown, arithmetized versions of these laws imply, under minimal conditions, an ω-inconsistent set of sentences.[31]

Ω-inconsistency is sometimes viewed as a decisive mark against a theory of truth. It seems to me, however, that this attitude rests on illicitly extending to the truth-concept ideas that are not appropriate for it. For example, it is of no consequence that a particular second-order version of an ω-inconsistent theory of truth possesses no standard model. This fact does not entail that the theory is trivial from a semantic point of view—that it semantically entails every sentence.[32] Such a conclusion would follow only if the truth-concept were required to possess a definite extension, but not otherwise. We have seen that sometimes the semantics of a circular concept is not characterizable through an assignment of extension. Sometimes the semantics requires an irreducible revision process.

Some thinkers, including Tarski, have claimed that the principles governing the concept of truth imply outright inconsistencies. I myself think that this claim is too strong. Nevertheless, something in its neighborhood is true: under certain conditions, the principles governing the concept of truth imply ω-inconsistencies. As far as I can see, this fact brings no ill effects in its wake. It is merely a reflection of the special character of the concept of truth—namely, its circularity—and the presence of a certain kind of infinitistic dependencies.[33]

Notes

[1]I shall be concerned in this essay only with non-contextual definitions. I am setting aside ostensive definitions, for example.

[2]This scheme is mentioned in passing on p. 185 of *RTT*. I, for one, did not appreciate its virtues when I was working on *RTT*.

[3]For definiteness, we may take $=, \bot, \neg, \&$, and $\forall$ as the primitive logical constants of $\mathscr{L}$, and we may take $\vee, \supset, \equiv$, and $\exists$ as defined. We shall sometimes use $\forall$ and $\exists$ as abbreviations for "for all" and "for some." Furthermore, we shall sometimes use symbols autonymously.

[4]The semantical scheme given below is easily extended to arbitrarily large interdependent systems. These systems may contain definitions that define terms belonging to diverse logical categories, including n-ary predicates and function symbols.

[5]If the double turnstile "$\models$" lacks a subscript, then the scheme in play should be understood to be **S**.

[6]With systems of interdependent definitions, the revision rule takes as input a hypothesis concerning the interpretation of *all* the defined terms and yields as output a possibly new hypothesis of the same sort.

[7]I follow common convention and use variables m, n, p to range over natural numbers and lower-case Greek letters α, β, and γ to range over ordinal numbers. I reserve δ, ρ and τ for revision rules.

[8]This treatment of limit stages follows the Belnap rule: no restriction is placed on whether or not an element unstable in a revision sequence $\mathscr{S}$ at a limit stage α belongs to $\mathscr{S}_\alpha$.

[9]The quantification over On-long sequences is dispensable; see *RTT*, part 5C.

[10]Saturated hypotheses were called "attractive" in *RTT*. A hypothesis h saturated [descending, recurring] for $\delta_{\mathscr{D},M}$ will also be said to be saturated [descending, recurring] for M relative to $\mathscr{D}$. Sometimes, if the context allows it, we suppress relative to $\mathscr{D}$ and/or M.

[11]Note that a revision sequence may need to go through many stages—indeed α many stages—before the cofinal hypothesis is reached.

[12]This is an instance of a general fact: for arbitrary $\mathscr{D}$ and M, if a unique hypothesis h is saturated for $\delta_{\mathscr{D},M}$ then "$\mathscr{D}$-validity in **S** relative to M" coincides with "truth in $M + h$."

[13]This definition is considered in *RTT*, Examples 5A.17-19.

[14]This is a consequence of the Belnap treatment of unstable elements at limit stages.

[15]See Shawn Standefer, "Non-Classical Circular Definitions."

[16]This scheme is named "$\mathbf{S}_0$" in *RTT*.

[17]For further information on finite definitions, see my "Finite Circular Definitions" and Maricarmen Martinez's "Some Closure Properties of Finite Definitions."

[18]I do not mean to suggest that the three desiderata are non-negotiable with respect to (e.g.) ostensive definitions.

[19]Scheme $\mathbf{S}^\sigma$ is modeled on the unrestricted supervaluational version of Saul Kripke's theory of truth; see Kripke, "Outline of a Theory of Truth."

[20]Kremer, "The Gupta-Belnap Systems $\mathbf{S}^\#$ and $\mathbf{S}^*$ Are Not Axiomatisable."

[21]Aldo Antonelli showed that the complexity is exactly $\mathbf{\Pi}^1_2$; see his "Complexity of Revision" and "Complexity of Revision, Revised."

[22]I have included the subscript 'r' (for 'revision') in '$\mathrm{DfI_r}$' to distinguish the present formulation of the rule from its traditional formulation. The same remarks applies to '$\mathrm{DfE_r}$' below.

[23]For a Fitch-style presentation of **C**, see *RTT*, part 5B.

[24]See also Standefer's "Solovay-Type Theorems for Circular Definitions."

[25]So, **C** is also complete with respect to this scheme, but this is not relevant to the point I am making.

[26]See my "On Circular Concepts," and Riccardo Bruni and Giacomo Sillari, "A Rational Way of Playing."

[27]See my "Finite Circular Definitions."

[28]Tarski, "Semantic Conception of Truth," §4.

[29]As before, I have introduced the subscript 'r' in '$\mathrm{TI_r}$' to distinguish the present formulation of the rule from its traditional formulation; similarly with '$\mathrm{TE_r}$'.

[30]Note that these laws cannot be proved using $\mathrm{TI_r}$ and $\mathrm{TE_r}$.

[31]McGee, "How Truth-Like Can a Predicate Be?" In fact, McGee shows that far weaker versions of the semantical laws suffice to generate ω-inconsistency. For a study of some ω-inconsistent theories of truth and their near relatives, see Volker Halbach, *Axiomatic Theories of Truth*, and Leon Horsten, et al., "Revision Revisited."

[32]Eduardo Barrio and Lavinia Picollo, "Notes on ω-Inconsistent Theories of Truth

in Second-Order Languages," pp. 738-739.

[33]Thanks to Stephen Mackereth and Shawn Standefer for their comments on an earlier draft of this essay.

References

[1] Antonelli, Aldo. 1994. "The Complexity of Revision." *Notre Dame Journal of Formal Logic* 35, 67–72.

[2] Antonelli, Aldo. 2002. "The Complexity of Revision, Revised." *Notre Dame Journal of Formal Logic* 43, 75–78.

[3] Barrio, Eduardo, & Picollo, Lavinia. 2013. "Notes on ω-Inconsistent Theories of Truth in Second-Order Languages." *Review of Symbolic Logic* 6, 733–741.

[4] Bruni, Riccardo, & Sillari, Giacomo. 2018. "A Rational Way of Playing: Revision Theory for Strategic Interaction." *Journal of Philosophical Logic* 47, 419-448.

[5] Gupta, Anil. 2000. "On Circular Concepts." In *Circularity, Definition, and Truth*, edited by André Chapuis and Anil Gupta, pp. 123–153. Indian Council of Philosophical Research, New Delhi. Also reprinted in Anil Gupta, *Truth, Meaning, Experience* (2011), pp. 95–134. Oxford University Press, Oxford.

[6] Gupta, Anil. 2006. "Finite Circular Definitions." In *Self-Reference*, edited by Thomas Bolander, Vincent F. Hendricks, and Stig Andur Pedersen, pp. 79–93. CSLI Publications, Stanford.

[7] Gupta, Anil, & Belnap, Nuel. 1993. *The Revision Theory of Truth.* MIT Press, Cambridge MA.

[8] Gupta, Anil, & Standefer, Shawn. 2017. "Conditionals in Theories of Truth." *Journal of Philosophical Logic* 46, 27–63.

[9] Halbach, Volker. 2011. *Axiomatic Theories of Truth.* Cambridge University Press, Cambridge.

[10] Horsten, Leon; Leigh, Graham E.; Leitgeb, Hannes; & Welch, Philip. 2012. "Revision Revisited." *Review of Symbolic Logic* 5, 642–665.

[11] Kremer, Philip. 1993. "The Gupta-Belnap Systems $\mathbf{S}^{\#}$ and $\mathbf{S}^{*}$ Are Not Axiomatisable." *Notre Dame Journal of Formal Logic* 34, 583–596.

[12] Kripke, Saul. 1975. "Outline of a Theory of Truth." *Journal of Philosophy* 72, 690–716.

[13] Martinez, Maricarmen. 2001. "Some Closure Properties of Finite Definitions." *Studia Logica* 68, 43–68.

[14] McGee, Vann. 1985. "How Truth-Like Can a Predicate Be? A Negative Result." *Journal of Philosophical Logic* 14, 399–410.

[15] Standefer, Shawn. 2015. "Solovay-Type Theorems for Circular Definitions." *Review of Symbolic Logic* 8, 467–487.

[16] Standefer, Shawn. 2017. "Non-Classical Circular Definitions." *Australasian Journal of Logic* 14, 147–180.

[17] Tarski, Alfred. 1944. "The Semantic Conception of Truth and the Foundations of Semantics." *Philosophy and Phenomenological Research* 4, 341–376.